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NONLINEAR RESPONSE OF AN IDEAL GAS BUBBLE TO AMBIENT PRESSURE CHANGE IN A QUIESCENT FLUID

Jong-Wook Ha and Seung-Man Yang†

Department of Chemical Engineering, Korea Advanced Institute of Science and Technology, 373-1 Kusung-dong, Yusung-ku, Taejon 305-701, Korea
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Abstract—This paper is concerned with the small-amplitude oscillations of a bubble composed of an ideal gas in response to an *abrupt* change in the ambient pressure field. Specifically, we consider the bubble response to a pressure pulse and a pressure step in an otherwise quiescent fluid. The method of analysis employed in the present study is a standard two-timing expansion to eliminate a secular behavior encountered in the asymptotic expansion. In the impulse response the secularity is self-induced due solely to the nonlinearity of the problem whereas the secularity in the step response arises from the change in the equilibrium bubble volume caused by the ambient pressure change. The two-timing solution for each response shows that the secularity modifies the natural frequency of the radial oscillation. Further, the critical intensity of either the pressure pulse or the pressure step for existence of the steady-state bubble radius is determined from the frequency modulated solution and the stability of the bubble response is also discussed in terms of the bubble compressibility and heat transfer across the interface.

Key words: Acoustic Pressure, Bubble Oscillation, Domain Perturbation, Monopole Emission, Potential Flow

INTRODUCTION

In this paper, we consider the nonlinear dynamics of bubble oscillations in response to abrupt changes in the ambient pressure in a fluid at rest at infinity. The bubble dynamics problems have attracted much attention for several reasons. First, the bubble motion coupled with the pressure variation in the surrounding is highly nonlinear, and second, this is responsible for many important effects, ranging from emulsifications to acoustic cavitation noise. The cavitation noise associated with the bubble oscillation is relevant particularly in the field of hydromachinery or to some aspects of the propulsion systems of submarines and other underwater vehicles. A time-dependent change of the bubble volume causes a periodic compression and expansion of the surrounding fluid, and thus produces sound wave. The energy exhausted by these waves is supplied from the kinetic energy of the bubble [1]. One of the most interesting subjects in cavitation noise is the modifications of frequency and amplitude by the shape and volume oscillations of the cavity. An important objective of the present analysis is to predict the amplitude and frequency modifications arising from the nonlinear effects on the bubble oscillation of the radial 'breathing' mode which is the most significant source of cavitation noise.

Since Rayleigh had first considered the problem of cavitation, the dynamics of a bubble under time-dependent pressure field has been extensively investigated and earlier studies of bubble dynamics and cavitation were well reviewed by Plesset and Prosperetti [2]. For the volume oscillation of a spherical bubble generated by the ambient pressure field, Rayleigh-Plesset equation is the most important governing equation. This equation describes the so called radial 'breathing' mode of bubble oscillations in an infinite viscous liquid, i.e., change in the bubble radius in response to perturbations in the ambient pressure. However, the Rayleigh-Plesset equation has some limitations for practical applications. For example, it does not include the effect of thermal damping. Moreover, when the gas contained in bubble is polytropic, the equation becomes highly nonlinear and exact solution is impossible for arbitrary amplitude of oscillations. In this case, analytic solution is possible only for small-amplitude motion in which the equation can be linearized [3]. In spite of these restrictions, the Rayleigh-Plesset equation is very useful in understanding the physics of bubble oscillations including the chaotic variation of the bubble radius and the resonance effects due to the self-induced secularity [4, 5]. Recently, studies about bubble have been extended to bubble oscillations either in the presence of external mean flow or in the electric field [6, 7]. Further, the nonlinear oscillations of a constant-volume bubble have also analyzed to examine the mechanism for energy transfer between modes of the shape oscillations [8].

When a bubble oscillates in response to perturbations of the ambient pressure, it executes the sustained growth and collapse in its volume. Further, when the bubble is immersed in an external mean flow or an electric field, the resulting nonuniform pressure on the bubble surface generates shape oscillations around an equilibrium shape. In the *absence* of the external field, the volume oscillation associated with the radial mode yields a monopole emission of sound which corresponds to a pressure disturbance decaying as 1/r with the distance r from the bubble. According to Minnaert [9], the frequency of the sound is close to that of the radial mode oscillations of a spherical bubble containing air. However, in the study of resonant interaction between shape and volume oscillations, Longuet-Higgins [10] showed that shape oscillations can also produce the monopole sound in a *quiescent* fluid when the frequency of radial mode is twice that of shape oscillations. Recently, Yang et al. [11] examined the resonant interaction for a bubble oscillating around a nonspherical equilib-

†To whom all correspondence should be addressed.

rium shape in the presence of an external field. In this case, the resonance occurs when the frequency of radial mode is either equal to or twice that of the shape oscillations.

In the present work, we are concerned with the nonlinear effects on bubble oscillation in a quiescent fluid due to abrupt changes in the ambient pressure. Since the fluid is at rest at infinity, the equilibrium shape of the bubble is spherical. Our work utilizes small-deformation, perturbation analysis carried to second and higher order in the amplitude of deformation, ε. Specifically, we will discuss the bubble dynamics in response to a pressure *pulse* at $t=0$ and a pressure *step* at $t=0$. We begin, at first, with formulating the governing equation and boundary conditions. In this part, we will perform dimensional analysis to determine the dimensionless parameters inherent in this problem and linearize the nonlinear problem by employing the domain perturbation technique. The present analysis identifies two different mechanisms for resonant interactions due to the secularities arising from nonlinearities of the original problem. The secularities will modify the oscillation frequency, which is relevant to the stability of the equilibrium bubble size.

FORMULATION OF THE PROBLEM

We begin by considering the governing equation and boundary conditions for a small-amplitude oscillation of a spherical bubble containing an ideal gas in response to an abrupt change in the ambient pressure. The surrounding fluid is incompressible Newtonian with viscosity μ and density ρ and assumed to be motionless in the absence of the pressure fluctuation. Thus, the velocity field is developed only by the disturbance due to the bubble oscillation. We assumed the surface of bubble to be characterized completely by a constant surface tension σ and the radius of bubble at the initial equilibrium state to be R_0. In the subsequent analysis, all the variables are nondimensionalized with the relevant characteristic length (l_c), time (t_c) and pressure (P_c) defined as

$$l_c = R_0,\ t_c = \left(\frac{R_0^3\rho}{\sigma}\right)^{1/2},\ P_c = \frac{\sigma}{R_0} \qquad (1)$$

Then, at equilibrium the difference of the pressures inside and outside of the bubble is balanced exactly by the surface tension, i.e.,

$$\tilde{P}_0 - P_0^{\infty} = 2 \qquad (2)$$

in which the tilde ($\sim$) symbolizes the pressure inside of the bubble. In this paper the subscript '0' denotes the equilibrium variables and the superscript '∞' the variables at large distances from the bubble. Eq. (2) is simply the dimensionless form of Laplace-Young equation.

We now define the oscillating surface of the bubble in response to perturbation in the ambient pressure as

$$S: r-1-f(t)=0 \qquad (3)$$

where f(t) represents the time-variation of the bubble radius. For the case in which the bubble contains only an insoluble gas and mass flux across the surface due to vaporization and condensation is small enough to be neglected, the radial component of velocity u_r can be related to the function f(t) by the kinematic boundary condition on the bubble surface.

$$u_r(r,t) = \frac{(1+f)^2}{r^2}\frac{df}{dt} \qquad (4)$$

The pressure field corresponding to the velocity field (4) is determined the Navier-Stokes equation as:

$$P(t) = P^{\infty}(t) + \frac{(1+f)^2}{r}\ddot{f} + 2\frac{(1+f)}{r}(\dot{f})^2 - \frac{(1+f)^4}{2r^4}(\dot{f})^2 \qquad (5)$$

in which $\dot{f} = df/dt$ and $\ddot{f} = d^2f/dt^2$. Since we are concerned with the spherical bubble, the dynamic boundary condition on the bubble surface $r=1+f$ is simply given by

$$\tilde{P}(t) - P_S(t) - \frac{4}{Re}\frac{\dot{f}}{(1+f)} = \frac{2}{(1+f)} \qquad (6)$$

where $\tilde{P}(t)$ denotes the pressure inside of the bubble and $P_S(t)$ the pressure on the surface outside of the bubble. The dimensionless parameter Re is the Reynolds number for bubble oscillations and defined as follows:

$$Re = \frac{\sqrt{\rho\sigma R_0}}{\mu}$$

The pressure $P_S(t)$ on the surface can be determined from (5) in terms of the ambient pressure $P^{\infty}(t)$ and the shape function f(t). Then, the dynamic condition (6) can be written as follows:

$$(1+f)\ddot{f} + \frac{3}{2}(\dot{f})^2 + \frac{4}{Re}\frac{\dot{f}}{(1+f)} = [\tilde{P}(t) - P^{\infty}(t)] - \frac{2}{(1+f)} \qquad (7)$$

The Reynolds number for a bubble executing shape oscillations is very large and the contributions from viscous forces are usually negligible. For example, when an air bubble of 100 μm in radius oscillates in water at 20℃, the Reynolds number is as large as Re = 85. Thus, in the analysis which follows we neglect the viscous terms from the governing equations.

In addition to the governing equation and boundary conditions for fluid motion, there exists a thermodynamic constraint with a pressure-volume relationship:

$$\tilde{P}(t) = (1+f)^{-3\gamma}\tilde{P}_0 \qquad (8)$$

Here, γ is a polytropic exponent which depends on the thermodynamic nature of the bubble oscillation. The exponent γ is bounded by the two limiting values. The lower limit corresponds to the slow oscillation case in which the rate of heat transfer is sufficiently fast that the temperature is uniform throughout the fluid including inside of the bubble. In this case, the oscillation is an isothermal process and the exponent γ is unity. The upper limit is for the fast oscillation in which the gas contained in the bubble is practically thermally insulated from the surrounding. In this case, γ is given by the ratio of the specific heats and has a value 1.4 for an ideal diatomic gas. For many situations of interest, the bubble behaves neither isothermally nor adiabatically, but somewhere in between two limits [12].

The problem defined above is a nonlinear free-boundary problem and analytic exact solution is not attainable for oscillations with an arbitrary amplitude. In this study, we consider small amplitude oscillations of a spherical gas bubble in response to 'abrupt' changes in the ambient pressure $P^{\infty}(t)$, i.e.,

$$P^{\infty}(t) = P_0^{\infty} + \varepsilon A(t) \qquad (9)$$

where $\varepsilon A(t)$ is the pressure perturbation from the equilibrium state and ε denotes the order of magnitude. The source of the oscillations in bubble volume is an abrupt change in the pressure at the bubble surface $\varepsilon A(t)$. This type of surface pressure can be produced experimentally via modulated ultrasonic acoustic wave fields [13]. Since we are interested in small amplitude oscillations for which analytic solution is possible, the magnitude of pressure perturbation is expected to be small, i.e., $\varepsilon \ll 1$. In this case, the magnitude of amplitude function f(t) is also $O(\varepsilon)$. Under these conditions, we can expand the thermodynamic relationship (8) as a Taylor series about $f=0$,

$$\frac{\tilde{P}}{\tilde{P}_0} = -3\gamma f + \frac{3}{2}\gamma(3\gamma+1)f^2 - \frac{1}{2}\gamma(1+3\gamma)(2+3\gamma)f^3 + O(f^4) \qquad (10)$$

Then, plugging (9) and (10) into (7) and carrying out the Taylor series expansion, we get

$$(1+f)\ddot{f} + \frac{3}{2}(\dot{f})^2 + \varepsilon \cdot A(t)$$
$$= -\tilde{P}_0\left[3\gamma f - \frac{3}{2}\gamma(3\gamma+1)f^2 + \frac{1}{2}\gamma(1+3\gamma)(2+3\gamma)f^3\right]$$
$$+2f - 2f^2 + 2f^3 + O(f^4) \qquad (11)$$

In obtaining the above equation, we utilized the equilibrium condition:

$$f_0 = 0$$
$$\tilde{P}_0 - P_0^{\infty} = 2$$

Thus, if the amplitude function f(t) for bubble radius is determined from (11), the velocity and pressure fields generated by the bubble oscillation can be obtained easily from (4) and (5), respectively. Since we consider the small amplitude oscillations, it is convenient to expand the amplitude function for the bubble radius in the asymptotic limit $\varepsilon \ll 1$.

$$f(t) = \sum_{n=1} \varepsilon^n f_n \qquad (12)$$

$$P_d(t) \equiv P(t) - P_0^{\infty} - \varepsilon \cdot A(t) = \sum_{n=1} \varepsilon^n P_n \qquad (13)$$

In (13), $P_d(t)$ is the disturbance pressure due to the bubble oscillation. As noted earlier, the monopole sound which is the most significant source of cavitation noise is related to the pressure fluctuation with decay like r^{-1}. Thus, we present here first term of the monopole pressure disturbance in terms of the amplitude function:

$$P_d^{\infty}(t) = \frac{\varepsilon}{r}\ddot{f}_1 + O(\varepsilon^2) \qquad (14)$$

Now, the amplitude function can be determined by substituting the asymptotic form (12) into (11). In the following sections, we evaluate the monopole pressure disturbance due to the bubble oscillation which is caused by the pressure impulse and the pressure step both applied at $t=0$ and we begin with the pressure impulse.

IMPULSE RESPONSE IN A QUIESCENT FLUID

We consider the radial mode oscillation of a bubble generated by an impulsive change in the ambient pressure. In this case, the perturbation $\varepsilon \cdot A(t)$ of the ambient pressure from the equilibrium state can be expressed in terms of Dirac delta function $\delta(t)$, i.e.,

$$A(t) = A_0\delta(t) \qquad (15)$$

Then, the solution for $f_n(t)(n=1, 2, \cdots)$ can be obtained straightforwardly from (11) and (15): The leading order problem is simply

$$\ddot{f}_1 + \omega^2 f_1 = -A_0\delta(t) \qquad (16)$$

in which the natural radian frequency ω of the radial oscillation with no contribution from the nonlinearity is defined as

$$\omega^2 \equiv 3\tilde{P}_0\gamma - 2 \qquad (17)$$

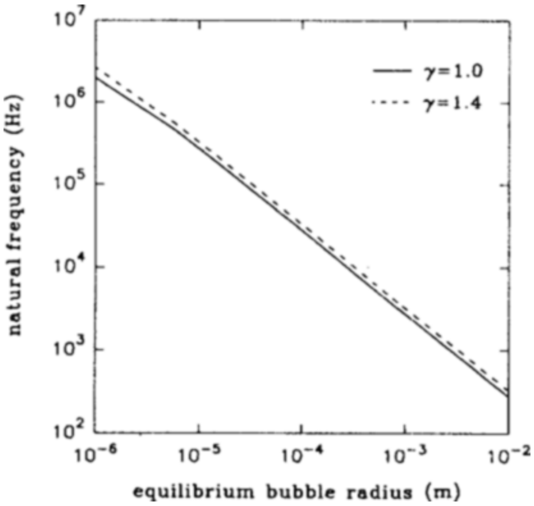

Fig. 1. Natural frequency $\omega/2\pi$ of the linear oscillation of radial mode as a function of the equilibrium radius R_0.

In Fig. 1, the dimensional natural frequency in Hz is plotted as a function of the equilibrium bubble radius for the polytropic constant $\gamma=1$ and 1.4. As anticipated, the natural frequency is increased as the bubble size becomes smaller. This clearly indicates that the smaller bubble is more stable than the larger bubble, which we shall see shortly. The leading order solution is given by

$$f_1 = -\frac{A_0}{\omega}\sin \omega t \qquad (18)$$

Similarly, the second order solution can be calculated by utilizing the leading order solution (18):

$$f_2 = C_1 \cos\omega t + C_2 \sin\omega t - \frac{A_0^2}{12\omega^4}[3(2+\gamma)\omega^2 + 2(3\gamma-1)]\cos 2\omega t$$
$$+\frac{A_0^2}{4\omega^4}[3\gamma\omega^2 + 2(3\gamma-1)] \qquad (19)$$

where the unknown integral constants C_1 and C_2 must be determined from the initial conditions for $O(\varepsilon^2)$ problem. It is clear from (19) that there is no secular behavior in the $O(\varepsilon^2)$ solution. In order to examine the existence of secularity, we seek the third order solution. The corresponding differential equation for $f_3(t)$ is given by:

$$\ddot{f}_3 + \omega^2 f_3 = -(f_2\ddot{f}_1 + f_1\ddot{f}_2 + 3\dot{f}_1\dot{f}_2) + \{\omega^2(1+3\gamma) + 2(3\gamma-1)\}f_1 f_2$$
$$+\left\{2 - \frac{1}{6}(\omega^2+2)(3\gamma+1)(3\gamma+2)\right\}f_1^3 \qquad (20)$$

Then, substituting f_1 and f_2 into (20), we encounter the *self-induced* secular terms violating the validity of regular perturbation method for a bounded solution as below:

$$\ddot{f}+\omega^2 f_3=-2\left(\frac{A_0}{\omega}\right)^3\left[\frac{\omega^2}{16}(6\gamma^2-3\gamma-2)+\frac{5}{12\omega^2}(3\gamma-1)^2+\frac{1}{8}(7\gamma-2)(3\gamma-1)\right]\cos\omega t+n.s.t's \quad (21)$$

in which *n.s.t's* denotes the nonsecular terms. Since we expect a bounded solution for a small perturbation in the ambient pressure, we should eliminate the *self-induced* secularities. The method of analysis for eliminating the secular behaviors is a typical multiple scale expansion. Details on the multiple-scale technique can be found in Bender and Orszag [14]. To do this, we introduce a new, slow time scale τ which is related to the fast time scale t by

$$\tau=\varepsilon^2 t \quad (22)$$

In the two-timing procedure, the $O(\varepsilon)$ solution is expressed in terms of two independent time scales, t and τ, that is,

$$f_1(t,\ \tau)=\frac{1}{2}[\alpha(\tau)\exp(i\omega t)]+\text{c.c.} \quad (23)$$

with

$$\alpha(0)=\frac{A_0}{\omega} \quad (24)$$

in which $\alpha(\tau)$ is the slowly varying amplitude function on the time scale τ and c.c. denotes the complex conjugate of the precedent terms. With this expression, we can solve the $O(\varepsilon^2)$ problem for f_2. The result is

$$f_2(t,\ \tau)=-\frac{1}{12\omega^2}[3(2+\gamma)\omega^2+2(3\gamma-1)]\alpha^2\exp(i2\omega t)+\frac{1}{8\omega^2}[3\gamma\omega^2+2(3\gamma-1)]\alpha\alpha^*+\frac{1}{2}\beta(\tau)\exp(i\omega t)+\text{c.c.} \quad (25)$$

where β is a slowly varying complex function of τ and can be determined in such a way to remove the secularity occurred in the higher order problem. In (25), α^* denotes the complex conjugate of α.

After substituting (23) and (25) into (20) and collecting secular terms, we obtain a differential equation for α, which can eliminate the secular terms:

$$\frac{d\alpha}{d\tau}=-\left[\frac{\omega}{16}(6\gamma^2-3\gamma-2)+\frac{5}{12\omega^3}(3\gamma-1)^2+\frac{1}{8\omega}(7\gamma-2)(3\gamma-1)\right]\alpha^2\alpha^* \quad (26)$$

The solution for α satisfying the initial condition (24) is given by

$$\alpha(\tau)=i\left(\frac{A_0}{\omega}\right)\exp\left\{-i\left(\frac{A_0}{\omega}\right)^2\left[\frac{\omega}{16}(6\gamma^2-3\gamma-2)+\frac{5}{12\omega^3}(3\gamma-1)^2+\frac{1}{8\omega}(7\gamma-2)(3\gamma-1)\right]\tau\right\} \quad (27)$$

Finally, the asymptotic solution for the amplitude function can be expressed in terms of the fast time scale t.

$$f_1(t)=-\left(\frac{A_0}{\omega}\right)\sin\left[\left[\omega-\frac{(\varepsilon A_0)^2}{16\omega}\left\{(6\gamma^2-3\gamma-2)+\frac{2}{\omega^2}(3\gamma-1)(7\gamma-2)+\frac{20(3\gamma-1)^2}{3\omega^4}\right\}\right]t\right] \quad (28)$$

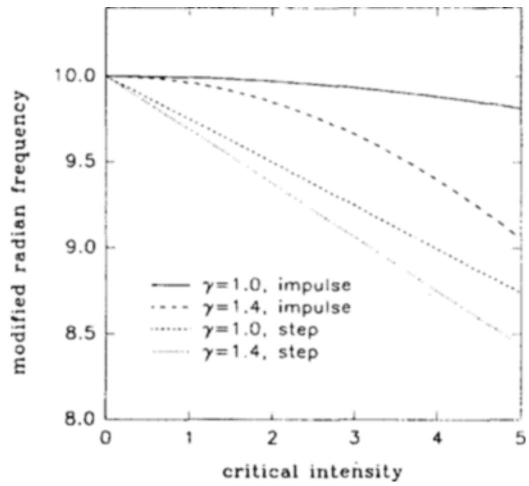

Fig. 2. Modified frequencies Ω as a function of the intensities of the pressure impulse and step for $\omega=10$.

It is simple matter to evaluate the monopole pressure associated with the bubble oscillation from (28). The result is given as:

$$P_d^{\infty}(t)=-\left(\frac{\varepsilon A_0\omega}{r}\right)\sin\left[\left[\omega-\frac{(\varepsilon A_0)^2}{16\omega}\left\{(6\gamma^2-3\gamma-2)+\frac{2}{\omega^2}(3\gamma-1)(7\gamma-2)+\frac{20(3\gamma-1)^2}{3\omega^4}\right\}\right]t\right] \quad (29)$$

Hence, the monopole pressure is in phase with the radial oscillation.

It can be easily seen from comparing (28) with (18) that the frequency is modified due to the nonlinear effects. In Fig. 2, the modified frequency is plotted as a function of the intensity of the pressure pulse. Also included for comparison is the modified frequency versus the intensity of the pressure step which we will discuss shortly in the next section. It can be easily seen from the figure that the frequency is decreased monotonically as the intensity of the pressure pulse becomes large, which is independent of γ and ω. Further, the rate of decrease is larger in the adiabatic oscillation than in the isothermal case. The fact that the frequency of oscillation decreases has an important physical significance, because at a critical intensity of the pressure pulse, the square of the true frequency of oscillation becomes zero and eigenvalues for the amplitude function change from pure imaginary to real. This critical intensity $(\varepsilon A_0)_{cr}$ will corresponds exactly to a limit point for existence of the steady state value for the bubble radius in the pressure pulse. The critical intensity can be determined readily from the present asymptotic solution. The result is

$$(\varepsilon A_0)_{cr}^{\ 2}=\frac{24\omega^6}{3\omega^4(6\gamma^2-3\gamma-2)+6\omega^2(3\gamma-1)(7\gamma-2)+20(3\gamma-1)^2} \quad (30)$$

In Fig. 3, the critical intensity is illustrated as a function of the polytropic constant γ for various values of the natural frequency ω. It can be easily seen that the bubble executing the isothermal oscillation is more stable than it would execute the adiabatic oscillation for the impulsive change in the ambient pressure. Thus, heat transfer across the interface enhances the stability. Further, as the radian natural frequency increases, the bubble becomes more stable. As noted earlier, the natural frequency is a decreasing function of the bubble radius and the larger bubble becomes

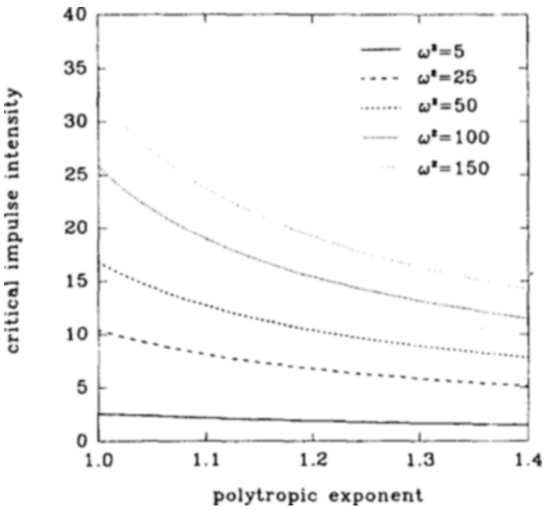

Fig. 3. Critical impulse intensity as a function of the polytropic exponent.

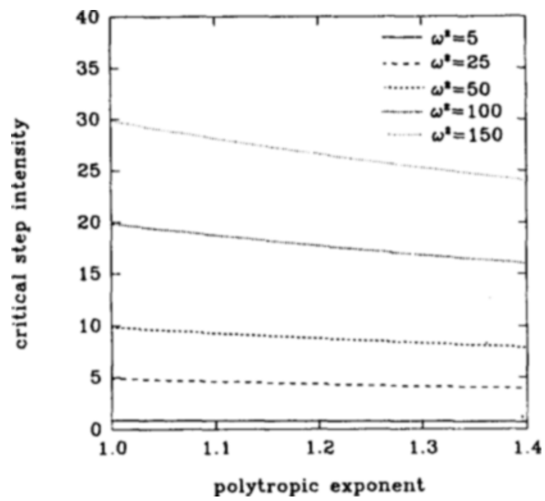

Fig. 4. Critical step intensity as a function of the polytropic exponent.

less stable.

STEP RESPONSE IN A QUIESCENT FLUID

In the preceding section, we have discussed the response of a bubble to an impulsive change in the ambient pressure. In this section, we consider the bubble response to a step change in the ambient pressure.

$$A(t)=A_0H(t) \tag{31}$$

where εA_0 is the intensity of the pressure step. In the step response, the leading order solution without taking into account the nonlinear effect is simply given by

$$f_1=-\frac{A_0}{\omega^2}(1-\cos\omega t) \tag{32}$$

Thus, the bubble radius oscillates around a new steady-state value $R=1-\varepsilon A_0/\omega^2$ in response to the pressure step. It can be noted that as the natural frequency is increased, the compressibility of the bubble is reduced.

Following the preceding analysis, we can easily show that the regular perturbation breaks down at $O(\varepsilon^2)$ at which a secularity appears. Thus, expecting a bounded solution of the bubble oscillation for a small perturbation in the ambient pressure, we define a new, slow time scale as:

$$\tau=\varepsilon t \tag{33}$$

Then, the leading order solution can be expressed in terms of the two time scales

$$f_1(t,\ \tau)=\frac{1}{2}\left[\alpha(\tau)\exp(i\omega t)-\frac{A_0}{\omega^2}\right]+\text{c.c.} \tag{34}$$

with

$$\alpha(0)=\frac{A_0}{\omega^2} \tag{35}$$

where, $\alpha(\tau)$ is a complex amplitude function and c.c. denotes the complex conjugate. The slowly varying function $\alpha(\tau)$ is determined in such a way that the secular terms at $O(\varepsilon^2)$ is eliminated so that the solution of $O(\varepsilon^2)$ problem remains bounded. The differential equation for $O(\varepsilon^2)$ problem can be obtained by utilizing the solution form (34) and given as:

$$\ddot{f}_2+\omega^2 f_2=-i\omega\dot{\alpha}(\tau)e^{i\omega t}-A_0\left(\frac{3\gamma+2}{2}+\frac{3\gamma-1}{\omega^2}\right)\alpha(\tau)e^{i\omega t} + \text{c.c.}+n.s.t.'s \tag{36}$$

The secularity at $O(\varepsilon^2)$ in this case is *not* self-induced but arises from the change in the steady-state bubble volume from the initial equilibrium state, which is caused by the ambient pressure change. As in the previous case, the condition for the absence of the secular behavior determines the slowly varying amplitude function which provides the solution for the bubble oscillation. The result can be expressed as follows:

$$f_1(t)=\frac{A_0}{\omega^2}\left[\cos\left[\omega+\frac{\varepsilon A_0}{\omega}\left\{\frac{3\gamma+2}{2}+\frac{1}{\omega^2}(3\gamma-1)\right\}\right]t-1\right] \tag{37}$$

Then, the monopole pressure associated with the bubble oscillation can be readily determined from (14) and (37):

$$P_d^{\infty}(t)=-\frac{\varepsilon A_0}{r}\left[\cos\left[\omega+\frac{\varepsilon A_0}{\omega}\left\{\frac{3\gamma+2}{2}+\frac{1}{\omega^2}(3\gamma-1)\right\}\right]t-1\right] \tag{38}$$

When $A_0>0$, the bubble volume decreases from the initial equilibrium state to a new steady-state value in the increased ambient pressure and the oscillation frequency increases. Consequently, the bubble is stabilized by the positive pressure step. When $A_0<0$, however, the response of the bubble is quite different from the response to the positive pressure step. For the negative pressure step, the new steady-state bubble volume in the reduced ambient pressure increases from the initial equilibrium value, which results in decrease in the oscillation frequency, which is depicted in Fig. 2. In fact, there is a critical intensity of the pressure step, $(-\varepsilon A_0)_c$ at which the square of the frequency is zero. As mentioned earlier, the critical intensity of the pressure step corresponds to the limit point for existence of the steady-state bubble volume in the reduced ambient pressure. The critical intensity can be determined readily from the present asymptotic solution (37) and given by

$$(-\varepsilon A_0)_c=\frac{\omega^4}{2(3\gamma-1)+\omega^2(2+3\gamma)} \tag{39}$$

Fig. 4 signifies the effects of the polytropic exponent γ and the natural frequency (or the compressibility) of the bubble on the critical intensity of the pressure step which leads to bubble breakup. It can be easily seen from the figure that the bubble executing the adiabatic oscillation is less stable than it would oscillates isothermally. Further, as the natural frequency increases, the bubble becomes more stable.

Finally, we consider the difference between the bubble responses to the abrupt changes and response to the oscillatory change in the ambient pressure, which has been considered by Leal [3]. In the oscillatory response, the ambient pressure is given by

$$A(t)=A_0 \sin \omega_0 t \tag{40}$$

where the forcing frequency ω_0 is arbitrary. In this case, provided only that ω_0 is not equal to ω, the asymptotic solutions obtained via a standard regular perturbation method are valid, i.e., the bubble radius oscillates periodically with an amplitude of $O(\varepsilon)$. When $\omega=\omega_0$, however, the secularity appears at $O(\varepsilon)$ due to the resonance with the oscillating pressure field. It means that, if a bounded solution is to exist, the $O(\varepsilon)$ pressure variation must be balanced by one or more of the nonlinear terms. This is quite different from the cases of the pressure pulse and pressure step in which the resonances occur at $O(\varepsilon^3)$ and $O(\varepsilon^2)$, respectively, due to either the *self-induced* secularity or the secularity arising from the equilibrium volume change. Leal considered the resonant interaction between the bubble and the ambient pressure and obtained a bounded solution by utilizing the two-timing expansion defined as:

$$f(t, \tau)=\varepsilon^{1/3}\lambda(\tau)\sin\{\omega t-\phi(\tau)\} \tag{41}$$

with

$$\tau=\varepsilon^{2/3}t \tag{42}$$

The response of a bubble to a periodic pressure field, in fact, differs from the results for the pressure pulse and the pressure step in several aspects. First, when the bubble oscillates in the oscillatory pressure field, the resonant interaction modifies not only the frequency but the amplitude of the radial oscillation. Second, the resonant interaction with the small amplitude pressure oscillation induces the bubble oscillation with an amplitude of $O(\varepsilon^{1/3})$ which is still asymptotically small but much larger than the $O(\varepsilon)$ amplitude of the pressure forcing. Furthermore, the disturbance pressure associated with the bubble oscillation is also $O(\varepsilon^{1/3})$ which is asymptotically very large compared to the pressure forcing. In the step or impulse response, the amplitudes of the bubble and disturbed pressure oscillations are the same order as the intensity of the pressure impulse or step. This is a consequence of the difference in the pressure forcings. In the pressure impulse or step, the pressure changes abruptly at $t=0$ and after then it remains constant without forcing any longer. In the oscillatory response, however, the pressure forcing is continuously sustained for $t\geq 0$.

CONCLUSION

The small-amplitude oscillations of a compressible bubble induced by a pressure pulse and pressure step in a quiescent fluid have been analyzed discussed using a standard method of multiple-scale analysis. From this analysis we have the following conclusions.

1. For an impulsive change in the ambient pressure, the regular perturbation gives the self-induced secular terms at $O(\varepsilon^3)$, which can be removed by the multiple-scale analysis. From the frequency-modulated solution at $O(\varepsilon)$, the critical intensity of the pressure impulse for existence of the steady-state bubble volume can be evaluated. The critical intensity is independent of the sign of the pressure impulse.

2. The bubble response to the step change in the ambient pressure contains the secularity at $O(\varepsilon^2)$ which modifies the oscillation frequency at $O(\varepsilon)$. For a positive pressure step, the bubble is always stable. However, for a negative pressure step in which the bubble oscillates in the reduced pressure, there exists a critical intensity of the step.

3. As the bubble compressibility increases or equivalently the natural frequency decreases, the bubble becomes less stable. Further, heat transfer across the interface enhances the stability in both the cases.

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